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ON THE SOLUTION OF PROBLEMS OF FILTRATION WITH A LIMIT GRADIENT*

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The plane stable filtration of an incompressible liquid with a limit gradient is considered /1/. A special non-linear filtration law is introduced, for which the basic system of equations obtained by transformation of the hodograph /2/ has a general solution which enables the theory of functions of a complex variable to be effectively employed. As a special case the proposed law contains the law considered in /3/. Solutions of the problems of the motion produced by a source in a narrow zone, and the motion from a source-sink pair are presented.

1. Let x and y be the rectangular Cartesian coordinates of a point in the plane of motion, $z=x+iy$ $(i=\sqrt{-1})$, v is the modulus of the filtration rate vector, θ is the angle of inclination of the filtration rate vector to the x axis, $\varphi=-H+\text{const}$, where H is the pressure head and ψ the stream function, and $\Phi(v)$ is the function that defines the filtration law.

The meaning of the functions ψ and $\Phi(v)$ can be defined by the differential relation

$$dz=e^{i\theta}\left[\frac{d\varphi}{\Phi(v)}+\frac{id\psi}{v}\right] \tag{1.1}$$

We will introduce into the analysis the function $\Phi(v)$ parametrically (parameter σ, $\sigma \geqslant 0$) by formulas

$$\Phi=\frac{\lambda m e^{\sigma}}{m+\operatorname{th} m\sigma},\quad v=\frac{\lambda m}{a}\frac{e^{\sigma}}{m+\operatorname{cth} m\sigma} \tag{1.2}$$

where λ, m and a are arbitrary positive constants.

Setting $\sigma=0$, we obtain $v=0, \Phi=\lambda$. We have, consequently, the law of filtration with a limit gradient. When $\sigma\to\infty$, then $v\to\infty, \Phi\to\infty$. Curves of Φ/λ against av/λ are shown in Fig.1 for various m, calculated from (1.2).

The case when $m=1, a=1$ was considered previously /3/. In that case

$$\Phi(v)=(v^2+\lambda^2)^{1/2} \tag{1.3}$$

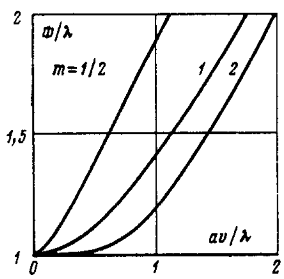

Fig.1

We shall assume φ and ψ to be functions of σ and θ. From the condition for the right-hand side of the differential relation (1.1) to be integrable we obtain the fundamental linear system

$$\frac{\partial\varphi}{\partial\theta}=a\operatorname{cth}^2 m\sigma\frac{\partial\psi}{\partial\sigma},\quad \frac{\partial\varphi}{\partial\sigma}=-a\operatorname{cth}^2 m\sigma\frac{\partial\psi}{\partial\theta} \tag{1.4}$$

whose general solution can be represented in the form /2/

$$\varphi=-a\operatorname{Re}[mF(\omega)-\operatorname{cth} m\sigma F'(\omega)]$$
$$\psi=\operatorname{Im}[mF(\omega)-\operatorname{th} m\sigma F'(\omega)] \tag{1.5}$$

where $F(\omega)$ is an arbitrary analytic function of the complex variable $\omega=\sigma+i\theta$.

An important stage in solving problms by the hodograph method is the transition to the physical plane of motion. Using (1.1), (1.2) and (1.5) we obtain the following convenient transition formula (C is an arbitrary constant):

$$\frac{\lambda}{a}\bar{z}=\frac{e^{-2\sigma}}{\operatorname{sh} 2m\sigma}\overline{e^{\omega}F'(\omega)}+\left(\frac{1}{m}+\operatorname{cth} 2m\sigma\right)e^{-\omega}F'(\omega)-\frac{m^2-1}{m}\int e^{-\omega}F'(\omega)\,d\omega+C \tag{1.6}$$

Note the function $F(\omega)=R\operatorname{ch} m\omega$, where R is a real constant. If $F(\omega)=R\operatorname{ch} m\omega$, then by (1.5) and (1.6), $\varphi\equiv 0, \psi\equiv 0, z\equiv\text{const}$. This function may prove useful in solving specific boundary value problems.

2. Below, we obtain a solution of the problem of a source in a narrow zone with impermeable boundaries when (1.2) applies. The solution of this problem for the law

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$$\Phi(v) = v + \lambda, \quad \lambda = \text{const} \tag{2.1}$$

appears in /5/ (see also /1/). In /3/ the solution was obtained when $m = 1, a = 1$ (law (1.3)).

Suppose a narrow strip of width $2l$ is bounded by the straight lines $x = l$ and $x = -l$. A source of intensity $4q$ is at the origin of coordinates. The modulus of the velocity at infinitely distant points of the narrow strip is denoted by v_1, and the corresponding value of σ is denoted by σ_1.

It is sufficient to investigate only the part of stream in the first quadrant. The half-strip $0 \leqslant \theta \leqslant \pi/2, \sigma \geqslant 0$ corresponds to this part of the flow in the ω plane. The boundary condition for the function $\psi(\sigma, \theta)$ is

$$\psi(\sigma, 0) = 0 \; (\sigma \geqslant 0), \quad \psi(0, \theta) = 0 \; (0 \leqslant \theta \leqslant \pi/2)$$
$$\psi(\sigma, \pi/2) = 0 \; (0 \leqslant \sigma < \sigma_1), \quad \psi(\sigma, \pi/2) = q \; (\sigma > \sigma_1)$$

We will solve the problem by the method analytic continuation /6/. For this we divide the half-strip $0 \leqslant \theta \leqslant \pi/2, \sigma \geqslant 0$ by a section of a straight line $\sigma = \sigma_1$ into two subregions: an infinite one and a finite one. In the first subregion we represent the function $F(\omega)$ in the form

$$F_1(\omega) = \frac{2q}{\pi}\left(A + \frac{\omega}{m} - \sum_{k=1}^{\infty} A_k e^{-2k\omega}\right) \tag{2.2}$$

and in the second subregion in the form

$$F_2(\omega) = \frac{2q}{\pi}\left(B + R \operatorname{ch} m\omega + \sum_{k=1}^{\infty} B_k \operatorname{ch} 2k\omega\right) \tag{2.3}$$

where A, B, R, A_k, B_k are real constants $(k = 1, 2, \ldots)$.

The boundary conditions for the function $\psi(\sigma, \theta)$ are in this case satisfied. Applying on line $\sigma = \sigma_1$ the condition of analytic continuation of the function $\psi(\sigma, \theta)$, we obtain

$$A_k = \frac{(-1)^k}{2k \operatorname{sh} m\sigma_1}\left[\frac{\operatorname{sh}(2k+m)\sigma_1}{2k+m} + \frac{\operatorname{sh}(2k-m)\sigma_1}{2k-m}\right] \tag{2.4}$$

$$B_k = \frac{(-1)^k e^{-2k\sigma_1}}{2k \operatorname{sh} m\sigma_1}\left[\frac{e^{-m\sigma_1}}{2k+m} + \frac{e^{m\sigma_1}}{2k-m}\right] \tag{2.5}$$

These formulas enable us to sum the series in expressions for $F_1'(\omega)$ and $F_2'(\omega)$. If we set

$$R = \frac{\pi \operatorname{cosec}(m\pi/2)}{2m \operatorname{sh} m\sigma_1} \tag{2.6}$$

we obtain the following solution, unique for the whole half-strip:

$$F(\omega) = \frac{2q}{\pi}\int_0^{\omega}\left[\frac{1}{m} + I_1(\omega) + I_2(\omega)\right] d\omega \tag{2.7}$$

$$I_1(\omega) = \frac{1}{2 \operatorname{sh} m\sigma_1}\int_{\omega+\sigma_1}^{\omega-\sigma_1} \frac{e^{m(\omega-t)}}{1+e^{2t}}\, dt$$

$$I_2(\omega) = \frac{1}{2 \operatorname{sh} m\sigma_1}\int_{\omega+\sigma_1}^{\omega-\sigma_1} \frac{e^{m(t-\omega)}}{1+e^{2t}}\, dt$$

Note that when m is rational, the integrals in the expressions for $I_1(\omega)$ and $I_2(\omega)$ may be represented in the form of finite combinations of elementary functions.

Using (1.6) after integration and reduction, we obtain the following connection between the coordinates of respective points in the planes ω and z:

$$\frac{\pi\lambda m\bar{z}}{2qa} = G(\sigma, \theta)e^{-\omega} - me^{-\omega} + (m + \operatorname{cth} m\sigma_1)e^{-\sigma_1} \operatorname{arc\,tg} e^{\sigma_1-\omega} + (m - \operatorname{cth} m\sigma_1)e^{\sigma_1} \operatorname{arctg} e^{-\sigma_1-\omega} \tag{2.8}$$

$$G(\sigma, \theta) = \frac{m}{\operatorname{sh} 2m\sigma}\left[\overline{I_1(\omega)} + \overline{I_2(\omega)} + e^{-2m\sigma}I_1(\omega) + e^{2m\sigma}I_2(\omega) + \frac{2}{m}\operatorname{ch}^2 m\sigma\right] \tag{2.9}$$

The constant C is determined by the condition $\bar{z} = 0$ when $\sigma = \infty$. The equations

$$\lim_{\sigma\to\infty} I_1(\sigma) = 0, \quad \lim_{\sigma\to\infty} I_2(\sigma) = 0$$

were used here; they can be obtained by applying the generalized integral theorem of the mean.

Let us find the boundary of the stagnation zone, for which we let σ in (2.8) tend to zero. The function in the expression in square brackets in (2.9) vanishes when $\sigma = 0$. Expanding the indeterminancy in (2.9) when $\sigma = 0$, we obtain

$$G(0,\theta) = \frac{1}{2}[I_1'(-i\theta) + I_2'(-i\theta) + I_1'(i\theta) + I_2'(i\theta) - 2mI_1(i\theta) + 2mI_2(i\theta)] = \frac{\operatorname{sh} 2\sigma_1}{\operatorname{th} m\sigma_1 (\operatorname{ch} 2\sigma_1 + \cos 2\theta)} \quad (2.10)$$

Moreover we have the equations

$$\operatorname{arc\,tg} e^{\sigma_1 - i\theta} = \frac{\pi}{2} - \operatorname{arc\,tg} e^{-\sigma_1 + i\theta} \quad (2.11)$$
$$\operatorname{arc\,tg} e^{-\sigma_1 + i\theta} = \frac{1}{2}\operatorname{arc\,tg}\frac{\cos\theta}{\operatorname{sh}\sigma_1} + \frac{i}{4}\ln\frac{\operatorname{ch}\sigma_1 + \sin\theta}{\operatorname{ch}\sigma_1 - \sin\theta}$$

Taking into account (2.10) and (2.11) we obtain the equation for the stagnation zone boundary in the form

$$z(\theta) = \frac{2qa}{\pi\lambda m}\Big[\frac{\pi}{2}(m + \operatorname{cth} m\sigma_1)e^{-\sigma_1} + (m\operatorname{sh}\sigma_1 - \operatorname{ch}\sigma_1 \operatorname{cth} m\sigma_1) \times \quad (2.12)$$
$$\operatorname{arctg}\frac{\cos\theta}{\operatorname{sh}\sigma_1} + \left(\frac{\operatorname{sh} 2\sigma_1 \operatorname{cth} m\sigma_1}{\operatorname{ch} 2\sigma_1 + \cos 2\theta} - m\right)e^{i\theta} + \frac{i}{2}(m\operatorname{ch}\sigma_1 - \operatorname{sh}\sigma_1 \operatorname{cth} m\sigma_1)\ln\frac{\operatorname{ch}\sigma_1 + \sin\theta}{\operatorname{ch}\sigma_1 - \sin\theta}\Big]$$

This proves the feasibility of defining the boundary of the stagnation zone using elementary functions for any m.

In the limiting case $\sigma_1 = 0$ $(v_1 = 0,\ l = \infty)$ we have

$$z(\theta) = \frac{2qa}{\pi\lambda m}\Big[\frac{\pi}{2}\left(m - \frac{1}{m}\right) + \frac{1}{m\cos\theta} + \quad (2.13)$$
$$\frac{1 - m^2\cos^2\theta}{m\cos^2\theta}e^{i\theta} + \frac{i}{2}\left(m - \frac{1}{m}\right)\ln\frac{1 + \sin\theta}{1 - \sin\theta}\Big]$$

If we set $m = 1, a = 1$, then from (2.12) and (2.13) we obtain the formulas given in /3/. Calculations were carried out using (2.12) including the equation

$$\frac{qa}{\lambda m} = \frac{le^{\sigma_1}}{m + \operatorname{cth} m\sigma_1}$$

which follows from the obvious equation $q = lv_1$. Parameter

$$b = \frac{av_1}{\lambda} = \frac{me^{\sigma_1}}{m + \operatorname{cth} m\sigma_1}$$

was introduced in the calculations.

The results obtained are shown in Fig.2 for $b = 0.2$ and several m. As can be seen from these results, the character of the original filtration law appreciably affects the size of the stagnation zone.

3. We shall now use the proposed filtration law to solve the problem of motion generated by the source-sink pair. An approximate solution of that problem when (2.1) holds is given in /7/ (see also /1/). The problem was also considered on the assumption that the function $\Phi(v)$ has the form (1.3)* . The method of solution chosen was different from that in the present paper.

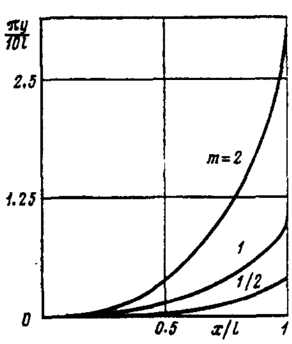

Fig.2

Suppose a source of intensity $2q$ is situated at a point with coordinates $x = l$, $y = 0$, and a sink of the same intensity is at a point with coordinates $x = -l$, $y = 0$ $(l > 0)$. The velocity at the origin of coordinates is denoted by v_0 and the corresponding value of σ is denoted by σ_0.

The half-strip $0 \leqslant \theta \leqslant \pi$, $\sigma \geqslant 0$ in the ω plane corresponds to motion in the first quadrant of the z plane. At the boundary of the half-plane we have the following condition:

$$\psi(\sigma, 0) = 0\ (\sigma \geqslant 0),\ \psi(0, \theta) = 0\ (0 \leqslant \theta \leqslant \pi)$$
$$\varphi(\sigma, \pi) = \text{const}\ (0 \leqslant \sigma \leqslant \sigma_0),\ \psi(\sigma, \pi) = q\ (\sigma \geqslant \sigma_0)$$

We take the solution of system (1.4) in the form

$$\varphi = -a\operatorname{Re}\Big[m\int_0^\infty W(\omega)\,d\omega - \operatorname{cth} m\sigma W(\omega)\Big] \quad (3.1)$$

* Pan'ko S.V., On certain plane steady filtration problems with limit gradient. Candidate Dissertation, Tomsk State University, Tomsk, 1975.

$$\psi = \operatorname{Im}\left[m\int_0^\omega W(\omega)\,d\omega - \operatorname{th} m\sigma W(\omega)\right]$$

where $W(\omega)$ is an arbitrary analytic function of the complex variable ω, $W(\omega) = F'(\omega)$.

The first three equations of the boundary condition written above are satisfied when the function $W(\omega)$ satisfies the following boundary conditions: $\operatorname{Im} W = 0$ when $\theta = 0, \sigma \geqslant 0$; $\operatorname{Re} W = 0$ when $\sigma = 0, 0 \leqslant \theta \leqslant \pi$; $\operatorname{Re} W = R \operatorname{sh} m\sigma$ when $\theta = \pi, 0 \leqslant \sigma \leqslant \sigma_0$, where R is an arbitrary constant; $\operatorname{Im} W = 0$ when $\theta = \pi, \sigma \geqslant \sigma_0$. Then on part of the boundary $\theta = \pi, \sigma \geqslant \sigma_0$ the function ψ takes some constant value. It will be shown below that the constant R can be determined so that one part of the boundary function ψ takes the required value q. Using the function $\zeta = \operatorname{ch}\omega$, we map the half-strip $0 \leqslant \theta \leqslant \pi$, $\sigma \geqslant 0$ onto the upper half-plane of the complex variable $\zeta = \xi + i\eta$. At the boundary $\eta = 0$ we have the condition

$$\begin{aligned}
&\operatorname{Im} W = 0\ (\xi \geqslant 1),\ \operatorname{Re} W = 0\ (-1 \leqslant \xi \leqslant 1)\\
&\operatorname{Re} W = R \operatorname{sh}(m \operatorname{Arch}(-\xi))\ (-\operatorname{ch}\sigma_0 \leqslant \xi \leqslant -1)\\
&\operatorname{Im} W = 0\ (\xi \leqslant -\operatorname{ch}\sigma_0)
\end{aligned}$$

Applying the Keldysh-Sedow formula /8, 9/, we obtain

$$W(\zeta) = \frac{1}{f(\zeta)}\left[\frac{R}{\pi i}\int_{-\operatorname{ch}\sigma_0}^{-1}\frac{u(t)f(t)}{t-\zeta}\,dt + W(\infty)\right] \tag{3.2}$$

$$u(t) = \operatorname{sh}(m \operatorname{Arch}(-t)), \qquad f(t) = \left(\frac{t-1}{t+\operatorname{ch}\sigma_0}\right)^{1/2}$$

The constant $W(\infty)$ is determined from the condition

$$\frac{R}{\pi i}\int_{-\operatorname{ch}\sigma_0}^{-1}\frac{u(t)f(t)}{t-1}\,dt + W(\infty) = 0 \tag{3.3}$$

and we arrive at the following expression for the function $W(\omega)$:

$$W(\omega) = R\left(\frac{\operatorname{ch}\omega + \operatorname{ch}\sigma_0}{\operatorname{ch}\omega - 1}\right)^{1/2}[J(\omega) - J(0)] \tag{3.4}$$

$$J(\omega) = \frac{1}{\pi i}\int_{-\operatorname{ch}\sigma_0}^{-1}\frac{u(t)f(t)}{t-\operatorname{ch}\omega}\,dt$$

Let σ_* be some fixed value of σ, and $\sigma_* > \sigma_0$. Using the second equation of (3.1), we obtain

$$\operatorname{Im}\left(\int_{\sigma_*}^{\sigma_*+\pi i} W(\omega)\,d\omega\right) = \frac{q}{m}$$

where as the integration path it is convenient take the segment of the straight line $\sigma = \sigma_*$. In the limit as $\sigma_* \to \infty$ we obtain

$$-\pi R J(0) = q/m$$

whence the required value $R = R_m$ is determined.

When m is an integer, the function $W(\omega)$ can be written is explicit form in terms of elementary functions

$$\begin{aligned}
&W(\omega) = W_1(\omega) - R_1 \operatorname{sh}\omega, \qquad W_1(\omega) = 2R_1 \operatorname{sh}\left(\frac{\omega}{2}\right)\varkappa(\omega)\\
&\varkappa(\omega) = \left(\operatorname{ch}^2\frac{\omega}{2} + \operatorname{sh}^2\frac{\sigma_0}{2}\right)^{1/2}, \quad R_1 = \frac{2q}{\pi(\operatorname{ch}\sigma_0 - 1)}, \quad m = 1\\
&W(\omega) = W_2(\omega) - R_2 \operatorname{sh} 2\omega,\\
&W_2(\omega) = 4R_2\left(\operatorname{ch}^2\frac{\sigma_0}{2} - 2\operatorname{ch}^2\frac{\omega}{2}\right)\operatorname{sh}\frac{\omega}{2}\,\varkappa(\omega)\\
&R_2 = \frac{2q}{\pi(1 + 3\operatorname{ch}\sigma_0)(\operatorname{ch}\sigma_0 - 1)}, \quad m = 2
\end{aligned}$$

The second terms in the expressions for $W(\omega)$ can be neglected. The functions $\varphi(\sigma, \theta)$, $\psi(\sigma, \theta)$ given by (3.1) satisfy the required boundary condition, if $W = W_1$ when $m = 1$ and $W = W_2$ when $m = 2$.

Let us consider the case of $m = 1$. In that case the formula for transition to the physical plane takes a very simple form

$$\frac{\lambda}{a}\bar{z} = \frac{e^{-\omega}}{\operatorname{sh} 2\sigma}\left[e^{2\sigma}W_1(\omega) + \overline{W_1(\omega)}\right] \tag{3.5}$$

where the constant C is determined from the condition $\bar{z} = 0$ when $\omega = \sigma_0 + \pi i$.

Setting $\sigma = \infty$, $\bar{z} = l$ in this formula we obtain

$$\lambda l = R_1 a \tag{3.6}$$

from which the dependence of σ_0 on the parameter $b = aq/(\lambda l)$ can be determined.

Passing to limit in (3.5) as $\sigma \to 0$ and using (3.6), we obtain for the stagnation zone boundary the equation

$$\frac{z(\theta)}{l} = \frac{e^{i\theta/2}}{4\varkappa(i\theta)}(\alpha_1 \cos\theta - i\alpha_2 \sin\theta + \alpha_3) \tag{3.7}$$

$$\alpha_1 = 3 - \operatorname{ch}\sigma_0, \quad \alpha_2 = 1 + \operatorname{ch}\sigma_0, \quad \alpha_3 = 3\operatorname{ch}\sigma_0 - 1$$

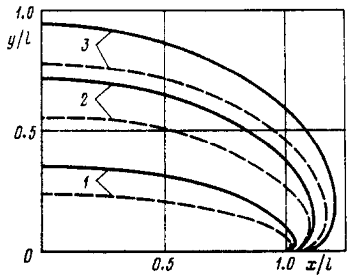

Fig.3

The case of $m = 2$ can be considered similarly. Evaluating the integral in (1.6) and transforming we obtain

$$\frac{\lambda}{a}\bar{z} = \frac{e^{-\omega}}{\operatorname{sh} 4\sigma}[e^{1\sigma} W_2(\omega) + \overline{W_2(\omega)}] + \frac{R_2}{2} e^{-\omega/2} s(\omega)\varkappa(\omega) + 6R_2\beta_0 \times$$

$$\ln \frac{\operatorname{ch}(\omega/2) + \varkappa(\omega)}{\operatorname{th}(\sigma_0/2)[\operatorname{sh}(\omega/2) + \varkappa(\omega)]}$$

$$s(\omega) = 4e^{\omega} + 1 + 2\operatorname{ch}\sigma_0 - 3\operatorname{ch}^2\sigma_0, \quad \beta_0 = \operatorname{sh}^2\frac{\sigma_0}{2}\operatorname{ch}^4\frac{\sigma_0}{2}$$

The connection between σ_0 and b when $m = 2$ is established using the equation

$$\lambda l = R_2 a \delta_0, \quad \delta_0 = 1 - 3\operatorname{sh}^4\frac{\sigma_0}{2} + 6\beta_0 \ln\left(\operatorname{cth}\frac{\sigma_0}{2}\right)$$

The equation of the stagnation zone boundary when $m = 2$ has the form

$$\frac{\delta_0 z(\theta)}{l} = 6\beta_0 \ln \frac{\cos(\theta/2) + \varkappa(i\theta)}{\operatorname{sh}(\sigma_0/2)} + 6i\beta_0 \operatorname{arctg}\frac{\sin(\theta/2)}{\varkappa(i\theta)} +$$

$$\frac{e^{i\theta/2}}{8\varkappa(i\theta)}(\beta_1\cos\theta + i\beta_2\sin\theta + \beta_3)$$

$$\beta_1 = 3 + 10\operatorname{ch}\sigma_0 - 9\operatorname{ch}^2\sigma_0, \quad \beta_2 = 1 - 2\operatorname{ch}\sigma_0 - 3\operatorname{ch}^2\sigma_0$$

$$\beta_3 = 1 + 9\operatorname{ch}^2\sigma_0 - 6\operatorname{ch}^3\sigma_0$$

The stagnation zone boundaries obtained when $m = 1$ (the solid lines) and $m = 2$ (the dashed lines) are shown in Fig.3, where curves *1–3* correspond to values of the parameter $b = aq/(\lambda l)$ equal to 0.1, 0.4, and 0.7.

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